Achieving the AWGN Channel Capacity With Lattice Gaussian Coding

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***Abstract*—We propose a new coding scheme using only one lattice that, under lattice decoding, achieves the $\frac{1}{2}\log(1+\mathsf{SNR})$ capacity of the additive white Gaussian noise (AWGN) channel, when the signal-to-noise ratio $\mathsf{SNR} > 3$. The scheme applies a discrete Gaussian distribution over an AWGN-good lattice, but does not require a shaping lattice or dither. Thus, it significantly simplifies the default lattice coding scheme of Erez and Zamir which additionally involves a quantization-good lattice. Using the flatness factor, we show that the error probability of the proposed scheme under minimum mean-square error (MMSE) lattice decoding is almost the same as that of Poltyrev's coding over an infinite lattice, for any rate up to the AWGN channel capacity.**

I. Introduction

A practical, structured code achieving the capacity of the power-constrained additive white Gaussian noise (AWGN) channel is the holy grail of communication theory. Lattice codes have been shown to hold this potential. Poltyrev initiated the study of lattice coding without a power constraint, which led to the notion of AWGN-good lattices [1]. Erez and Zamir dealt with the issue of the power constraint using nested lattice codes, where a quantization-good lattice serves as the shaping lattice while the AWGN-good lattice serves as the coding lattice [2]. Despite these significant progresses, major obstacles persist from a practical point of view. The scheme of [2] not only requires a dither which complicates the implementation, but also the construction of a quantization-good lattice nested with an AWGN-good lattice is not solved, to the best of our knowledge.

In this paper, we resolve such issues by employing *lattice Gaussian coding* when the signal-to-noise ratio $\mathsf{SNR} > 3$. More precisely, the codebook has a discrete Gaussian distribution over an AWGN-good lattice (so the remaining problem is the construction of AWGN-good lattices, which is beyond the scope of this paper though). Intuitively, since only shaping is lacking in Poltyrev's technique, the probabilistic shaping inherent with lattice Gaussian distribution will enable it to achieve the AWGN channel capacity.

Earlier, non-uniform signaling using discrete Gaussian inputs was respectively used in [3, 4] for shaping over the AWGN channel and in [5] for semantic security over the Gaussian wiretap channel. The new contribution of this paper is to use the flatness factor [5] to show that discrete Gaussian signaling over AWGN-good lattices can approach the capacity of the power-constrained Gaussian channel by using minimum mean-square error (MMSE) lattice decoding. The proposed approach enjoys a few salient features. Firstly, throughout the paper, we do not not use a shaping lattice. Secondly, in contrast to what is nowadays the common practice of lattice coding [2], we do not use a dither. These will simplify the implementation of the system.

After the draft of this paper was submitted, we became aware of Zamir's independent work using non-uniform signaling [6], where achieving the AWGN channel capacity was posed as a major open question. This paper serves as an answer to [6] in the affirmative. Also, it was asked in [6] whether dithering is really necessary. Although Forney pointed out earlier that dither is unnecessary, it was still needed to decouple MMSE and lattice decoding [7]. In contrast, MMSE arises naturally in maximum-a-posteriori (MAP) decoding for the proposed lattice Gaussian coding scheme, without resorting to dithering.

As we will see, the lattice Gaussian distribution behaves like the continuous Gaussian distribution in many aspects, while still preserving the rich structures of a lattice. Since the continuous Gaussian distribution is capacity achieving for many problems in information theory, we expect the lattice Gaussian distribution will find more applications, especially in network information theory.

Throughout this paper, we use the natural logarithm, denoted by $\log$, and information is measured in nats.

II. Lattice Coding and Lattice Gaussian Distribution

In this section, we introduce the mathematical tools we will need to describe and analyze the proposed coding scheme.

A. Preliminaries of Lattice Coding

An n-dimensional lattice Λ in the Euclidean space $\mathbb{R}^n$ is a set defined by

$$\Lambda = \mathcal{L}(\mathbf{B}) = \{\mathbf{Bx} : \mathbf{x} \in \mathbb{Z}^n\}$$

where the columns of the basis matrix $\mathbf{B} = [\mathbf{b}_1 \cdots \mathbf{b}_n]$ are linearly independent.

For a vector $\mathbf{x} \in \mathbb{R}^n$, the nearest-neighbor quantizer associated with Λ is $Q_\Lambda(\mathbf{x}) = \arg\min_{\lambda\in\Lambda} \|\lambda - \mathbf{x}\|$. We define the modulo lattice operation by $\mathbf{x} \mod \Lambda \triangleq \mathbf{x} - Q_\Lambda(\mathbf{x})$. The

Voronoi cell of Λ, defined by $\mathcal{V}(\Lambda) = \{\mathbf{x} : Q_\Lambda(\mathbf{x}) = \mathbf{0}\}$, specifies the nearest-neighbor decoding region.

The theta series of Λ (see, e.g., [8]) is defined as

$$\Theta_\Lambda(q) = \sum_{\lambda\in\Lambda} q^{\|\lambda\|^2} \tag{1}$$

where $q = e^{j\pi z}$ ($\Im(z) > 0$). Letting z be purely imaginary, and assuming $\tau = \Im(z) > 0$, we can alternatively express the theta series as

$$\Theta_\Lambda(\tau) = \sum_{\lambda\in\Lambda} e^{-\pi\tau\|\lambda\|^2}. \tag{2}$$

Let us also introduce the notion of lattices which are good for the Gaussian channel without power constraint:

Definition 1 (AWGN-good)**.** *Given $\varepsilon > 0$ and an n-dimensional lattice Λ, let W^n be an i.i.d. Gaussian random vector of variance σ_ε^2 such that $\mathbb{P}\{\mathsf{W}^n \notin \mathcal{V}(\Lambda)\} = \varepsilon$. Consider the corresponding volume-to-noise ratio (VNR) $\gamma_\Lambda(\sigma_\varepsilon) = \frac{(V(\Lambda))^{\frac{2}{n}}}{\sigma_\varepsilon^2}$. The sequence of lattices $\Lambda^{(n)}$ is* AWGN-good *if, for all $\varepsilon \in (0,1)$,*

$$\lim_{n\to\infty} \gamma_{\Lambda^{(n)}}(\sigma_\varepsilon) = 2\pi e$$

and if, for a fixed VNR greater than $2\pi e$, the quantity

$$\mathbb{P}\{\mathsf{W}^n \notin \mathcal{V}(\Lambda)\}$$

vanishes in n.

Erez and Zamir [2] showed that lattice coding and decoding can achieve the capacity of the Gaussian channel. More precisely, one can prove the existence of a sequence of nested lattices $\Lambda_s^{(n)} \subset \Lambda_f^{(n)}$ such that

- the shaping lattice $\Lambda_s^{(n)}$ is quantization-good,
- the fine lattice $\Lambda_f^{(n)}$ is AWGN-good.

When a random dither at the transmitter and an MMSE filter at the receiver are used, the Voronoi signal constellation $\Lambda_f^{(n)} \cap \mathcal{V}(\Lambda_s^{(n)})$ approaches the capacity of the mod-$\Lambda_s^{(n)}$ Gaussian channel, and consequently the capacity of the Gaussian channel, when n is large (see [2]).

B. Lattice Gaussian Distribution

For $\sigma > 0$ and $\mathbf{c} \in \mathbb{R}^n$, we define the Gaussian distribution of variance σ centered at $\mathbf{c} \in \mathbb{R}^n$ as

$$f_{\sigma,\mathbf{c}}(\mathbf{x}) = \frac{1}{(\sqrt{2\pi}\sigma)^n} e^{-\frac{\|\mathbf{x}-\mathbf{c}\|^2}{2\sigma^2}},$$

for all $\mathbf{x} \in \mathbb{R}^n$. For convenience, we write $f_\sigma(\mathbf{x}) = f_{\sigma,\mathbf{0}}(\mathbf{x})$.

We also consider the Λ-periodic function

$$f_{\sigma,\Lambda}(\mathbf{x}) = \sum_{\lambda\in\Lambda} f_{\sigma,\lambda}(\mathbf{x}) = \frac{1}{(\sqrt{2\pi}\sigma)^n} \sum_{\lambda\in\Lambda} e^{-\frac{\|\mathbf{x}-\lambda\|^2}{2\sigma^2}}, \tag{3}$$

for all $\mathbf{x} \in \mathbb{R}^n$. Observe that $f_{\sigma,\Lambda}$ restricted to the quotient $\mathbb{R}^n/\Lambda$ is a probability density.

We define the *discrete Gaussian distribution* over Λ centered at $\mathbf{c} \in \mathbb{R}^n$ as the following discrete distribution taking values in $\lambda \in \Lambda$:

$$D_{\Lambda,\sigma,\mathbf{c}}(\lambda) = \frac{f_{\sigma,\mathbf{c}}(\lambda)}{f_{\sigma,\mathbf{c}}(\Lambda)}, \quad \forall\lambda \in \Lambda,$$

where $f_{\sigma,\mathbf{c}}(\Lambda) \triangleq \sum_{\lambda\in\Lambda} f_{\sigma,\mathbf{c}}(\lambda)$. Again for convenience, we write $D_{\Lambda,\sigma} = D_{\Lambda,\sigma,\mathbf{0}}$.

It will be useful to define the discrete Gaussian distribution over a coset of Λ, i.e., the shifted lattice $\Lambda - \mathbf{c}$:

$$D_{\Lambda-\mathbf{c},\sigma}(\lambda - \mathbf{c}) = \frac{f_\sigma(\lambda - \mathbf{c})}{f_{\sigma,\mathbf{c}}(\Lambda)}, \quad \forall\lambda \in \Lambda.$$

Note the relation $D_{\Lambda-\mathbf{c},\sigma}(\lambda - \mathbf{c}) = D_{\Lambda,\sigma,\mathbf{c}}(\lambda)$, namely, they are a shifted version of each other.

C. Flatness Factor

The flatness factor of a lattice Λ quantifies the maximum variation of $f_{\sigma,\Lambda}(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^n$.

Definition 2 (Flatness factor [5])**.** *For a lattice Λ and for a parameter σ, the flatness factor is defined by:*

$$\epsilon_\Lambda(\sigma) \triangleq \max_{\mathbf{x}\in\mathcal{R}(\Lambda)} |V(\Lambda) f_{\sigma,\Lambda}(\mathbf{x}) - 1|$$

where $\mathcal{R}(\Lambda)$ is a fundamental region.

In other words, $f_{\sigma,\Lambda}(\mathbf{x})$ is within $1 \pm \epsilon_\Lambda(\sigma)$ from the uniform distribution over $\mathcal{R}(\Lambda)$.

Proposition 1 (Expression of $\epsilon_\Lambda(\sigma)$ [5])**.** *We have:*

$$\epsilon_\Lambda(\sigma) = \left(\frac{\gamma_\Lambda(\sigma)}{2\pi}\right)^{\frac{n}{2}} \Theta_\Lambda\left(\frac{1}{2\pi\sigma^2}\right) - 1.$$

Consider the ensemble of mod-p lattices (Construction A) [9]. For integer $p > 0$, let $\mathbb{Z}^n \to \mathbb{Z}_p^n : \mathbf{v} \mapsto \overline{\mathbf{v}}$ be the element-wise reduction modulo-p. The mod-p lattices are defined as $\Lambda_C \triangleq \{\mathbf{v} \in \mathbb{Z}^n : \overline{\mathbf{v}} \in C\}$, where p is a prime and C is a linear code over $\mathbb{Z}_p$. Quite often, scaled mod-p lattices $a\Lambda_C \triangleq \{a\mathbf{v} : \mathbf{v} \in \Lambda_C\}$ for some $a \in \mathbb{R}^+$ are used. The fundamental volume of such a lattice is $V(a\Lambda_C) = a^n p^{n-k}$, where n and k are the block length and dimension of the code C, respectively.

The following result, derived from the Minkowski-Hlawka theorem, guarantees the existence of sequences of mod-p lattices whose flatness factors can vanish as $n \to \infty$.

Theorem 1 ([5])**.** *$\forall\sigma > 0$ and $\forall\delta > 0$, there exists a sequence of mod-p lattices $\Lambda^{(n)}$ such that*

$$\epsilon_{\Lambda^{(n)}}(\sigma) \le (1+\delta) \cdot \left(\frac{\gamma_{\Lambda^{(n)}}(\sigma)}{2\pi}\right)^{\frac{n}{2}}, \tag{4}$$

i.e., the flatness factor goes to zero exponentially for any fixed VNR $\gamma_{\Lambda^{(n)}}(\sigma) < 2\pi$.

D. Properties of the Flatness Factor

In this subsection we collect properties of the flatness factor that will be useful in the paper. In a nutshell, the flatness factor serves two purposes. On one hand, it makes the folded distribution (3) flat; on the other hand, it smooths the discrete Gaussian distribution.

From the definition of the flatness factor, one can derive the following result:

Lemma 1. *For all $\mathbf{c} \in \mathbb{R}^n$ and $\sigma > 0$, we have:*

$$f_{\sigma,\mathbf{c}}(\Lambda) \in [1 - \epsilon_\Lambda(\sigma), 1 + \epsilon_\Lambda(\sigma)] \frac{1}{V(\Lambda)}.$$

The following result shows that the variance per dimension of the discrete Gaussian $D_{\Lambda,\sigma,\mathbf{c}}$ is not too far from σ^2 when the flatness factor is small [5].

Lemma 2. *Let L be sampled from the Gaussian distribution $D_{\Lambda,\sigma,\mathbf{c}}$. If $\varepsilon \triangleq \epsilon_\Lambda\left(\sigma/\sqrt{\frac{\pi}{\pi-1}}\right) < 1$, then*

$$\left| \mathbb{E}\left[\|\mathsf{L} - \mathbf{c}\|^2 \right] - n\sigma^2 \right| \le \frac{2\pi\varepsilon}{1-\varepsilon}\sigma^2.$$

From the maximum-entropy principle [10, Chap. 11], it follows that the discrete Gaussian distribution maximizes the entropy given the average energy and given the same support over a lattice. This is still so even if we restrict the constellation to a finite region of a lattice. The following lemma further shows that if the flatness factor is small, the entropy rate of a discrete Gaussian $D_{\Lambda,\sigma,\mathbf{c}}$ is almost equal to the differential entropy of a continuous Gaussian vector of variance σ^2 per dimension, minus $\log V(\Lambda)$, that of a uniform distribution over the fundamental region of Λ.

Lemma 3 (Entropy of discrete Gaussian [5])**.** *Let $\mathsf{L} \sim D_{\Lambda,\sigma,\mathbf{c}}$. If $\varepsilon \triangleq \epsilon_\Lambda\left(\sigma/\sqrt{\frac{\pi}{\pi-1}}\right) < 1$, then the entropy of L satisfies*

$$\left| \mathbb{H}(\mathsf{L}) - \left[n\log(\sqrt{2\pi e}\sigma) - \log V(\Lambda) \right] \right| \le \varepsilon',$$

where $\varepsilon' = -\log(1-\varepsilon) + \frac{\pi\varepsilon}{1-\varepsilon}$.

III. Lattice Gaussian Codebook And Error Probability

A. The Proposed Scheme

Now we describe the proposed coding scheme based on the lattice Gaussian distribution for the AWGN channel with power constraint P. The SNR is given by $\mathsf{SNR} = P/\sigma^2$ for noise variance σ^2. Let L be an AWGN-good lattice of dimension n_L. For the sake of generality, let the codebook be $L - \mathbf{c}$, where $\mathbf{c}$ is a proper shift as is often the case for various reasons in practice [11]. The encoder maps the information bits to points in $L - \mathbf{c}$, which obey the lattice Gaussian distribution $D_{L-\mathbf{c},\sigma_0}$:

$$D_{L-\mathbf{c},\sigma_0}(\mathbf{x}) = \frac{\frac{1}{(\sqrt{2\pi}\sigma_0)^{n_L}} e^{-\frac{\|\mathbf{x}\|^2}{2\sigma_0^2}}}{f_{\sigma_0,\mathbf{c}}(L)}, \quad \mathbf{x} \in L - \mathbf{c}.$$

We assume the flatness factor is small, under certain conditions to be made precise in the following. Particularly, this means that the transmission power of this scheme $P \to \sigma_0^2$.

Since the lattice points are not equally probable a priori in the lattice Gaussian signaling, we will use MAP decoding. The following connection was proven in [5] for the special case $\mathbf{c} = 0$. For completeness, we extend the proof to the general case.

Proposition 2 (Equivalence between MAP decoding and MMSE lattice decoding)**.** *Let $\mathbf{x} \sim D_{L-\mathbf{c},\sigma_0}$ be the input signaling of an AWGN channel where the noise variance is σ^2 per dimension. Then MAP decoding is equivalent to Euclidean lattice decoding of $L - \mathbf{c}$ using a scaling coefficient $\alpha = \frac{\sigma_0^2}{\sigma_0^2+\sigma^2}$, which is asymptotically equal to the MMSE coefficient $\frac{P}{P+\sigma^2}$.*

Proof: The received signal is given by $\mathbf{y} = \mathbf{x} + \mathbf{w}$, where $\mathbf{x} \in L - \mathbf{c}$ and $\mathbf{w}$ is the i.i.d. Gaussian noise vector of variance σ^2. Thus the MAP decoding metric is given by

$$\begin{aligned}
\mathbb{P}(\mathbf{x}|\mathbf{y}) &= \frac{\mathbb{P}(\mathbf{x},\mathbf{y})}{\mathbb{P}(\mathbf{y})} \propto \mathbb{P}(\mathbf{y}|\mathbf{x})\mathbb{P}(\mathbf{x}) \\
&\propto \exp\left(-\frac{\|\mathbf{y}-\mathbf{x}\|^2}{2\sigma^2} - \frac{\|\mathbf{x}\|^2}{2\sigma_0^2} \right) \\
&\propto \exp\left(-\frac{1}{2}\left(\frac{\sigma_0^2+\sigma^2}{\sigma_0^2\sigma^2} \left\| \frac{\sigma_0^2}{\sigma_0^2+\sigma^2}\mathbf{y} - \mathbf{x} \right\|^2 \right) \right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\arg\max_{\mathbf{x}\in L-\mathbf{c}} \mathbb{P}(\mathbf{x}|\mathbf{y}) &= \arg\min_{\mathbf{x}\in L-\mathbf{c}} \left\| \frac{\sigma_0^2}{\sigma_0^2+\sigma^2}\mathbf{y} - \mathbf{x} \right\|^2 \\
&= \arg\min_{\mathbf{x}\in L-\mathbf{c}} \|\alpha\mathbf{y} - \mathbf{x}\|^2 \qquad (5)
\end{aligned}$$

where $\alpha = \frac{\sigma_0^2}{\sigma_0^2+\sigma^2}$ is known, thanks to Lemma 2, to be asymptotically equal to the MMSE coefficient $\frac{P}{P+\sigma^2}$. □

Therefore, the MAP decoder is simply given by

$$\hat{\mathbf{x}} = Q_{L-\mathbf{c}}(\alpha\mathbf{y})$$

where $Q_{L-\mathbf{c}}$ denotes, in a similar fashion to Q_L, the minimum Euclidean decoder for shifted lattice $L - \mathbf{c}$.

B. Error Probability

Let us analyze the average error probability of the MAP decoder. Suppose $\mathbf{x} \in L - \mathbf{c}$ is sent. The received signal after MMSE scaling can be written as

$$\mathbf{y} = \alpha(\mathbf{x} + \mathbf{w}) = \mathbf{x} + (\alpha - 1)\mathbf{x} + \alpha\mathbf{w}.$$

The decoding error probability associated with $\mathbf{x}$ is given by

$$\begin{aligned}
P_e(\mathbf{x}) &= 1 - \int_{\mathbf{x}+\mathcal{V}(L)} \frac{1}{(\sqrt{2\pi}\alpha\sigma)^{n_L}} \exp\left\{ -\frac{\|\mathbf{y}-\alpha\mathbf{x}\|^2}{2\alpha^2\sigma^2} \right\} d\mathbf{y} \\
&= 1 - \int_{\mathcal{V}(L)} \frac{1}{(\sqrt{2\pi}\alpha\sigma)^{n_L}} \exp\left\{ -\frac{\|\mathbf{y}-(\alpha-1)\mathbf{x}\|^2}{2\alpha^2\sigma^2} \right\} d\mathbf{y} \\
&= \int_{\overline{\mathcal{V}}(L)} \frac{1}{(\sqrt{2\pi}\alpha\sigma)^{n_L}} \exp\left\{ -\frac{\|\mathbf{y}-(\alpha-1)\mathbf{x}\|^2}{2\alpha^2\sigma^2} \right\} d\mathbf{y}
\end{aligned}$$

$$\sum_{\mathbf{x}\in L-\mathbf{c}} \exp\left\{-\frac{\|\mathbf{y}+\mathbf{x}\|^2}{2\frac{\sigma_0^4}{\sigma_0^2+\sigma^2}}\right\} \in \left[1-\epsilon_L\left(\frac{\sigma_0^2}{\sqrt{\sigma_0^2+\sigma^2}}\right), 1+\epsilon_L\left(\frac{\sigma_0^2}{\sqrt{\sigma_0^2+\sigma^2}}\right)\right]\left(\sqrt{2\pi}\frac{\sigma_0^2}{\sqrt{\sigma_0^2+\sigma^2}}\right)^{n_L}\frac{1}{V}. \tag{7}$$

$$\begin{aligned}
P_e &\in \left[\frac{1-\epsilon_L\left(\frac{\sigma_0^2}{\sqrt{\sigma_0^2+\sigma^2}}\right)}{1+\epsilon_L(\sigma_0)}, \frac{1+\epsilon_L\left(\frac{\sigma_0^2}{\sqrt{\sigma_0^2+\sigma^2}}\right)}{1-\epsilon_L(\sigma_0)}\right]\frac{1}{(2\pi\alpha\sigma_0\sigma)^{n_L}}\left(\sqrt{2\pi}\frac{\sigma_0^2}{\sqrt{\sigma_0^2+\sigma^2}}\right)^{n_L}\int_{\overline{\mathcal{V}}(L)}\exp\left\{-\frac{\|\mathbf{y}\|^2}{2\frac{\sigma_0^2\sigma^2}{\sigma_0^2+\sigma^2}}\right\}d\mathbf{y} \\
&= \left[\frac{1-\epsilon_L\left(\frac{\sigma_0^2}{\sqrt{\sigma_0^2+\sigma^2}}\right)}{1+\epsilon_L(\sigma_0)}, \frac{1+\epsilon_L\left(\frac{\sigma_0^2}{\sqrt{\sigma_0^2+\sigma^2}}\right)}{1-\epsilon_L(\sigma_0)}\right]\frac{1}{\left(\sqrt{2\pi}\frac{\sigma_0\sigma}{\sqrt{\sigma_0^2+\sigma^2}}\right)^{n_L}}\int_{\overline{\mathcal{V}}(L)}\exp\left\{-\frac{\|\mathbf{y}\|^2}{2\frac{\sigma_0^2\sigma^2}{\sigma_0^2+\sigma^2}}\right\}d\mathbf{y} \\
&\overset{(a)}{\to} \frac{1}{(\sqrt{2\pi}\frac{\sigma_0\sigma}{\sqrt{\sigma_0^2+\sigma^2}})^{n_L}}\int_{\overline{\mathcal{V}}(L)}\exp\left\{-\frac{\|\mathbf{y}\|^2}{2\frac{\sigma_0^2\sigma^2}{\sigma_0^2+\sigma^2}}\right\}d\mathbf{y} \\
&\overset{(b)}{=} \frac{1}{(\sqrt{2\pi}\tilde{\sigma})^{n_L}}\int_{\overline{\mathcal{V}}(L)}\exp\left\{-\frac{\|\mathbf{y}\|^2}{2\tilde{\sigma}^2}\right\}d\mathbf{y}.
\end{aligned} \tag{8}$$

where $\overline{\mathcal{V}}(L)$ denote the complement of the Voronoi region $\mathcal{V}(L)$ in $\mathbb{R}^{n_L}$.

The average decoding probability is given by

$$\begin{aligned}
P_e &= \sum_{\mathbf{x}\in L-\mathbf{c}} \frac{\frac{1}{(\sqrt{2\pi}\sigma_0)^{n_L}}e^{-\frac{\|\mathbf{x}\|^2}{2\sigma_0^2}}}{f_{\sigma_0,\mathbf{c}}(L)}P_e(\mathbf{x}) \\
&= \sum_{\mathbf{x}\in L-\mathbf{c}} \frac{\frac{1}{(\sqrt{2\pi}\sigma_0)^{n_L}}e^{-\frac{\|\mathbf{x}\|^2}{2\sigma_0^2}}}{f_{\sigma_0,\mathbf{c}}(L)}\times \\
&\quad \int_{\overline{\mathcal{V}}(L)}\frac{1}{(\sqrt{2\pi}\alpha\sigma)^{n_L}}\exp\left\{-\frac{\|\mathbf{y}-(\alpha-1)\mathbf{x}\|^2}{2\alpha^2\sigma^2}\right\}d\mathbf{y} \\
&= \frac{\frac{1}{(2\pi\alpha\sigma_0\sigma)^{n_L}}}{f_{\sigma_0,\mathbf{c}}(L)}\times \\
&\quad \sum_{\mathbf{x}\in L-\mathbf{c}}\int_{\overline{\mathcal{V}}(L)} e^{-\frac{\|\mathbf{x}\|^2}{2\sigma_0^2}}\exp\left\{-\frac{\|\mathbf{y}-(\alpha-1)\mathbf{x}\|^2}{2\alpha^2\sigma^2}\right\}d\mathbf{y} \\
&= \frac{\frac{1}{(2\pi\alpha\sigma_0\sigma)^{n_L}}}{f_{\sigma_0,\mathbf{c}}(L)}\times \\
&\quad \sum_{\mathbf{x}\in L-\mathbf{c}}\int_{\overline{\mathcal{V}}(L)}\exp\left\{-\frac{\frac{\sigma_0^2}{\sigma^2}\|\mathbf{y}\|^2+\|\mathbf{y}+\mathbf{x}\|^2}{2\frac{\sigma_0^4}{\sigma_0^2+\sigma^2}}\right\}d\mathbf{y} \\
&= \frac{\frac{1}{(2\pi\alpha\sigma_0\sigma)^{n_L}}}{f_{\sigma_0,\mathbf{c}}(L)}\times \\
&\quad \int_{\overline{\mathcal{V}}(L)}\exp\left\{-\frac{\|\mathbf{y}\|^2}{2\frac{\sigma_0^2\sigma^2}{\sigma_0^2+\sigma^2}}\right\}\sum_{\mathbf{x}\in L-\mathbf{c}}\exp\left\{-\frac{\|\mathbf{y}+\mathbf{x}\|^2}{2\frac{\sigma_0^4}{\sigma_0^2+\sigma^2}}\right\}d\mathbf{y}.
\end{aligned} \tag{6}$$

Now the key observation is that, by Lemma 1, the infinite sum over L within the above integral is almost a constant for any $\mathbf{y}$ and any $\mathbf{c}$, as shown in (7) at the top of this page.

Substituting (7) back into (6), and noting that $f_{\sigma_0,\mathbf{c}}(L) \in [1-\epsilon_L(\sigma_0), 1+\epsilon_L(\sigma_0)]\frac{1}{V}$, we derive the expression of P_e as shown in (8) at the top of this page, where (a) holds under the conditions $\epsilon_L\left(\frac{\sigma_0^2}{\sqrt{\sigma_0^2+\sigma^2}}\right)\to 0$ and $\epsilon_L(\sigma_0)\to 0$, and in (b) we define $\tilde{\sigma} \triangleq \frac{\sigma_0\sigma}{\sqrt{\sigma_0^2+\sigma^2}}$.

But (8) is just the error probability of standard lattice decoding for noise variance $\tilde{\sigma}^2$. Since L is good for AWGN, P_e will vanish if

$$V(L)^{2/n_L} > 2\pi e\tilde{\sigma}^2. \tag{9}$$

Moreover, it means that the average error probability is almost the same as that of Poltyrev [1], including error exponent etc. (with σ^2 replaced by $\tilde{\sigma}^2$). More precisely, the error probability is bounded by

$$P_e \le e^{-n_L E_P(\mu)}. \tag{10}$$

In the previous formula, $E_P(\mu)$ denotes the Poltyrev exponent

$$E_P(\mu) = \begin{cases} \frac{1}{2}\left[(\mu-1)-\log\mu\right] & 1<\mu\le 2 \\ \frac{1}{2}\log\frac{e\mu}{4} & 2\le\mu\le 4 \\ \frac{\mu}{8} & \mu\ge 4 \end{cases} \tag{11}$$

where $\mu = \frac{\gamma_L(\tilde{\sigma})}{e}$.

We need to satisfy the conditions $\epsilon_L\left(\frac{\sigma_0^2}{\sqrt{\sigma_0^2+\sigma^2}}\right)\to 0$ and $\epsilon_L(\sigma_0)\to 0$. Obviously, the first condition subsumes the second one. So, for mod-p lattices, we can satisfy it by making

$$\gamma_L\left(\frac{\sigma_0^2}{\sqrt{\sigma_0^2+\sigma^2}}\right) = \frac{V(L)^{2/n_L}}{\frac{\sigma_0^4}{\sigma_0^2+\sigma^2}} < 2\pi, \tag{12}$$

Conditions (9) and (12) are compatible if

$$\sigma_0^2 > e\sigma^2 \tag{13}$$

which is a very mild condition, i.e, the SNR is larger than e.

C. Rate

Now, to satisfy the volume constraint (9), we choose the fundamental volume $V(L)$ such that $V(L)^{2/n_L} = 2\pi e\tilde{\sigma}^2(1+\varepsilon'')$ for some small $\varepsilon'' \to 0$.

By Lemma 2, we have

$$\sigma_0^2 \geq \frac{1}{1+\frac{2\pi\varepsilon}{n_L(1-\varepsilon)}}P, \tag{14}$$

where $\varepsilon = \epsilon_L\left(\sigma_0/\sqrt{\frac{\pi}{\pi-1}}\right) < 1$.

By Lemma 3, the rate of the code is given by

$$\begin{aligned} R &\geq \log(\sqrt{2\pi e}\sigma_0) - \frac{1}{n_L}\log V(L) - \frac{\varepsilon'}{n_L} \\ &= \log(\sqrt{2\pi e}\sigma_0) - \frac{1}{2}\log\left(2\pi e\frac{\sigma_0^2\sigma^2}{\sigma_0^2+\sigma^2}\right) - \\ &\qquad \frac{1}{2}\log(1+\varepsilon'') - \frac{\varepsilon'}{n_L} \\ &\geq \frac{1}{2}\log\left(1+\frac{\sigma_0^2}{\sigma^2}\right) - \frac{1}{2}\varepsilon'' - \frac{\varepsilon'}{n_L} \end{aligned}$$

where ε' is as defined in Lemma 3. Meanwhile, (14) leads to

$$\begin{aligned} \frac{1}{2}\log\left(1+\frac{\sigma_0^2}{\sigma^2}\right) &\geq \frac{1}{2}\log\left(1+\frac{\mathsf{SNR}}{1+\frac{2\pi\varepsilon}{n_L(1-\varepsilon)}}\right) \\ &\geq \frac{1}{2}\log\left(1+\mathsf{SNR}\right) - \frac{1}{2}\log\left(1+\frac{2\pi\varepsilon}{n_L(1-\varepsilon)}\right) \\ &\geq \frac{1}{2}\log\left(1+\mathsf{SNR}\right) - \frac{\pi\varepsilon}{n_L(1-\varepsilon)}. \end{aligned}$$

Thus,

$$\begin{aligned} R &\geq \frac{1}{2}\log\left(1+\mathsf{SNR}\right) - \frac{\pi\varepsilon}{n_L(1-\varepsilon)} - \frac{1}{2}\varepsilon'' - \frac{\varepsilon'}{n_L} \\ &\to \frac{1}{2}\log\left(1+\mathsf{SNR}\right) \end{aligned} \tag{15}$$

if $\varepsilon < 1$ and $\varepsilon'' \to 0$. It can be verified that (9) and $\varepsilon < 1$ are compatible if $\mathsf{SNR} > \frac{\pi}{\pi-1}e - 1 \approx 3$.

Therefore, using this lattice Gaussian codebook, we can achieve a rate arbitrarily close to the channel capacity while making the error probability vanish exponentially. We summarize the main results in the following theorem:

Theorem 2 (Coding theorem for the lattice Gaussian codebook)**.** *Consider a lattice code whose codewords are drawn from the discrete Gaussian distribution $D_{L-\mathbf{c},\sigma_0}$ for an AWGN-good lattice L. If $\mathsf{SNR} > 3$, then any rate (15) up to the channel capacity $\frac{1}{2}\log(1+\mathsf{SNR})$ is achievable, while the error probability of MMSE lattice decoding vanishes exponentially fast as in (10).*

Remark 1. It is possible to improve the threshold $\mathsf{SNR} > 3$. It is likely to be e.

IV. Comparison with Voronoi Constellations

Finally we compare with Voronoi constellations or nested lattice codes where the shaping lattice is good for quantization [2]. In such a scheme, the transmitted signal (subject to a random dither) is uniformly distributed on the Voronoi region of the shaping lattice. It is shown in [12] that such a uniform distribution converges to a Gaussian distribution in a weak sense, that is, the *normalized* Kullback-Leibler divergence (i.e., divided by the dimension) tends to zero. Since the Voronoi region of a quantization-good lattice converges to a sphere, the peak power is $n_L P$ asymptotically for average power P.

Our proposed scheme uses a discrete Gaussian distribution over L, hence requiring neither shaping nor dithering. Since it uses the entire lattice, the peak power seems to be infinite. Nevertheless, this need not be the case. By [13], if $\mathbf{x} \sim D_{L-\mathbf{c},\sigma_0}$, we have

$$\mathbb{P}(\|\mathbf{x}\| > \sqrt{2\pi n_L}\sigma_0) < \frac{1+\epsilon_L(\sigma_0)}{1-\epsilon_L(\sigma_0)}2^{-n_L}. \tag{16}$$

Therefore, as long as $\epsilon_L(\sigma_0)$ is bounded by a constant, the outer points need not to be sent, and the sent points can be drawn from a sphere of radius $\sqrt{2\pi n_L}\sigma_0$. The peak power is $2\pi n_L P$ asymptotically, which is larger than that of the Voronoi constellation by a factor 2π. Thus, in this aspect, the lattice Gaussian codebook is not much different from a finite constellation.

Acknowledgments

This work was supported in part by FP7 project PHYLAWS.

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