Lattices and Physical-Layer Security: A finite-dimensional analysis

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Parts of these results have been obtained in the framework of PHYLAWS project

Part I

Introduction

Outline

The Gaussian Wiretap Channel

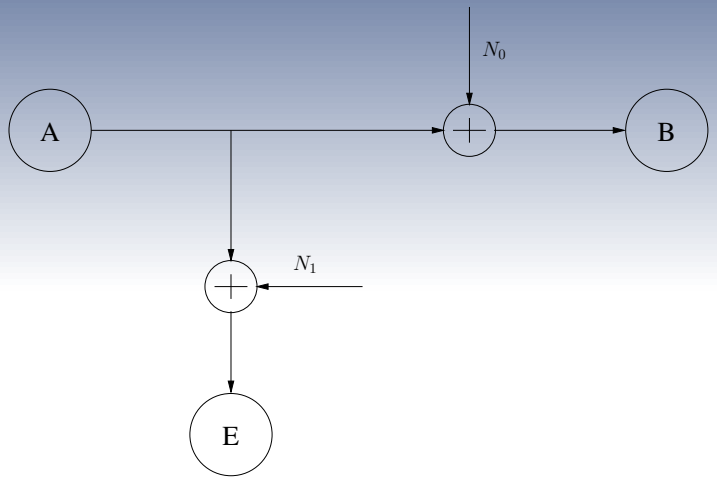

Figure : The Gaussian Wiretap Channel model

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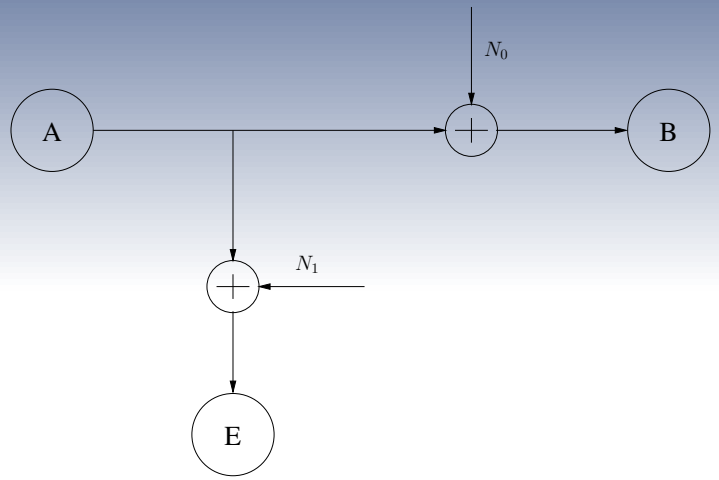

Figure : The Gaussian Wiretap Channel model

The secrecy capacity is given by

$$C_s = [C_{A\to B} - C_{A\to E}]^+$$

where $C_{A\to B} = \log_2\left(1+\frac{P}{N_0}\right)$ and $C_{A\to E} = \log_2\left(1+\frac{P}{N_1}\right)$ can be achieved by doing **lattice coding**.
Of course, $C_s > 0$ if $N_0 < N_1$.

Encoder Design

- The problem of Wiretap is a problem of **labelling** transmitted symbols with data bits

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+2 mod (4) Channel

We suppose the alphabet $\mathbb{Z}_4$ and a channel Alice$\hookrightarrow$Eve that outputs

$$y = x + 2$$

with probability $1/2$ and x with same probability. The **symbol** error probability is $1/2$.

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Confidential data must be encoded through b_1. On b_0, put random bits.

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Uniform Noise

Assume that **Alice** → **Eve** channel is corrupted by an additive uniform noise

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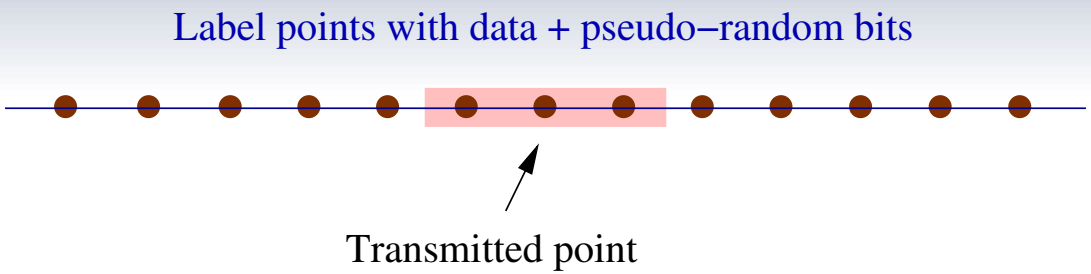

Figure : Constellation corrupted by uniform noise

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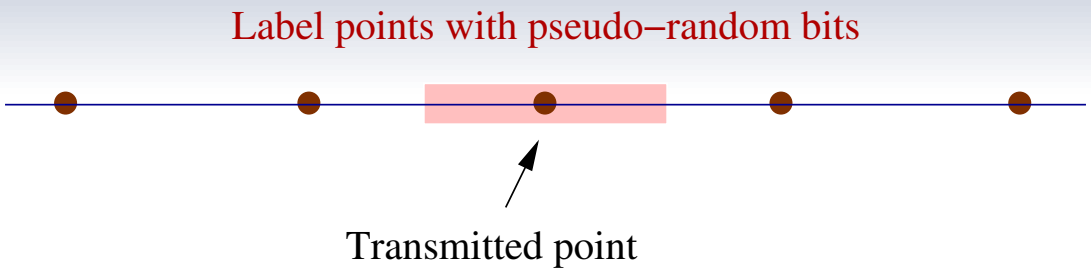

Figure : Points can be decoded **error free**: label with pseudo-random symbols

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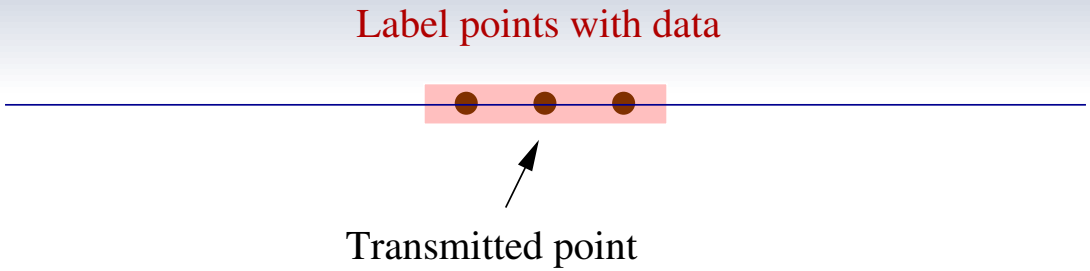

Figure : Points are **not distinguishable**: label with data

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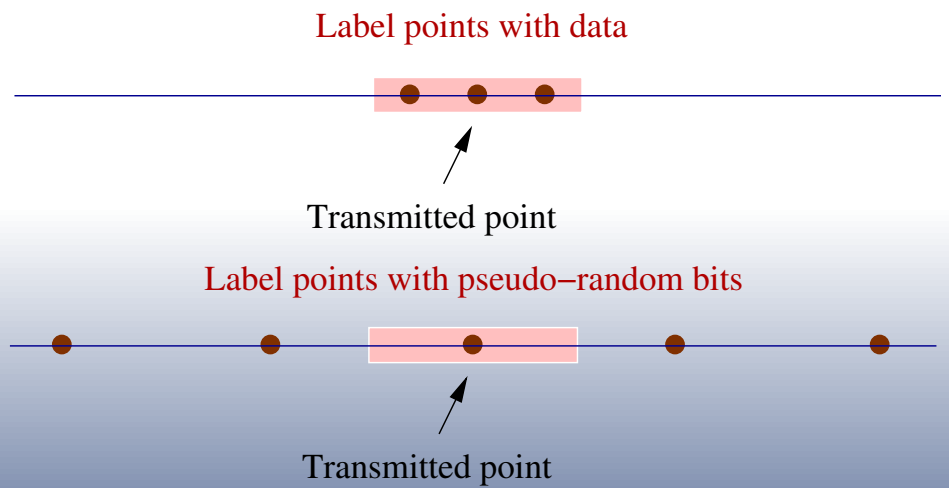

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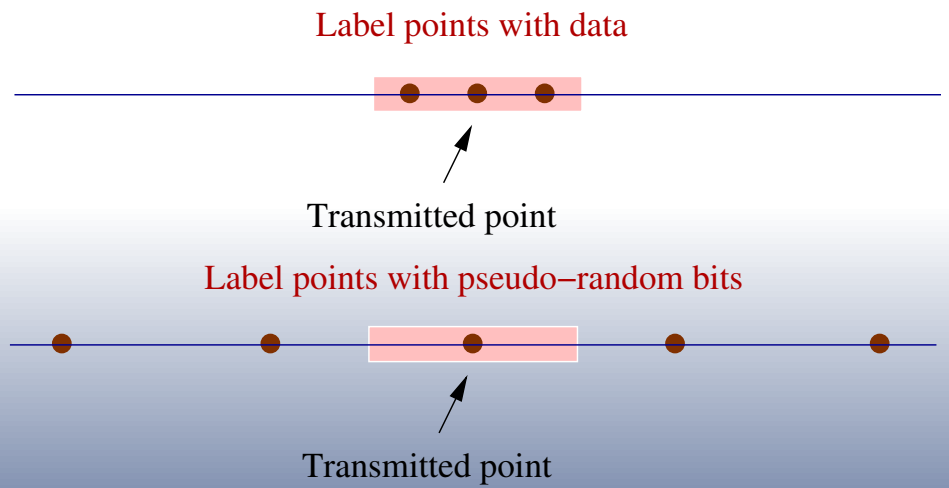
Error Probability

Pseudo-random symbols are perfectly decoded by Eve when data error probability will be high.

- unfortunately **not valid** for **Gaussian** noise.



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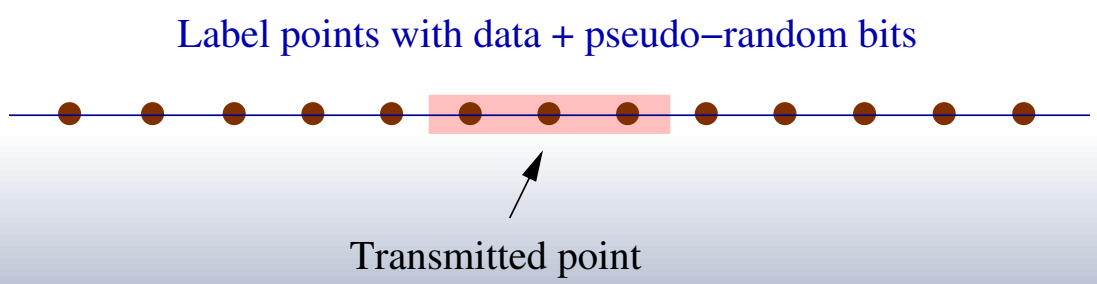
Coset Coding with Integers



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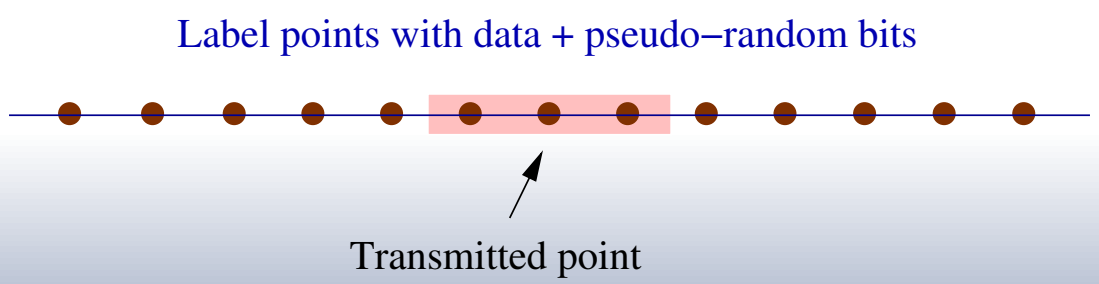
Coset Coding with Integers

Example

- Suppose that points x are in $\mathbb{Z}$.
- Euclidean division

$$x = 3q + r$$

- q carries the pseudo-random symbols while r carries the data **or** "pseudo-random symbols label points in $3\mathbb{Z}$ while data label elements of $\mathbb{Z}/3\mathbb{Z}$".



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Lattice Coset Coding

Gaussian noise is **not** bounded: it **needs** a $n-$dimensional approach (then let $n \to \infty$ for **sphere hardening**).

	$1-$dimensional	$n-$dimensional
Transmitted lattice	$\mathbb{Z}$	Fine lattice Λ_b
Pseudo-random symbols	$m\mathbb{Z} \subset \mathbb{Z}$	Coarse lattice $\Lambda_e \subset \Lambda_b$
Data	$\mathbb{Z}/m\mathbb{Z}$	Cosets Λ_b/Λ_e

Table : From the example to the general scheme

Part II

Secrecy Gain and Flatness Factor

Outline

Eve's Probability of Correct Decision (data)

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Can Eve decode the data?

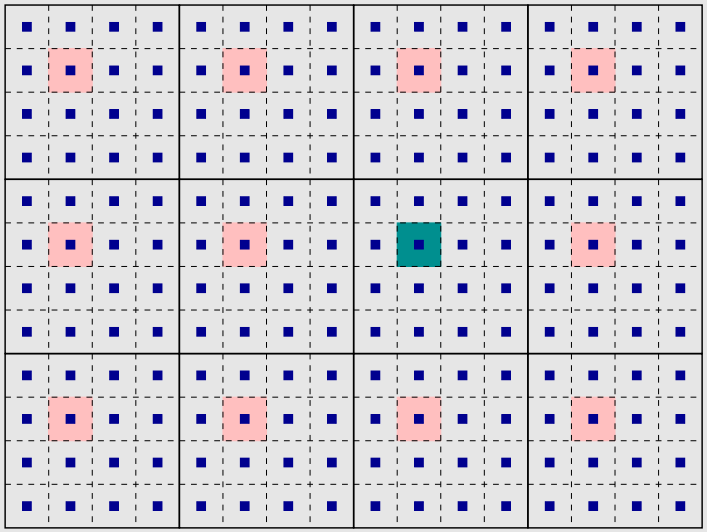

Figure : Eve correctly decodes when finding another coset representative

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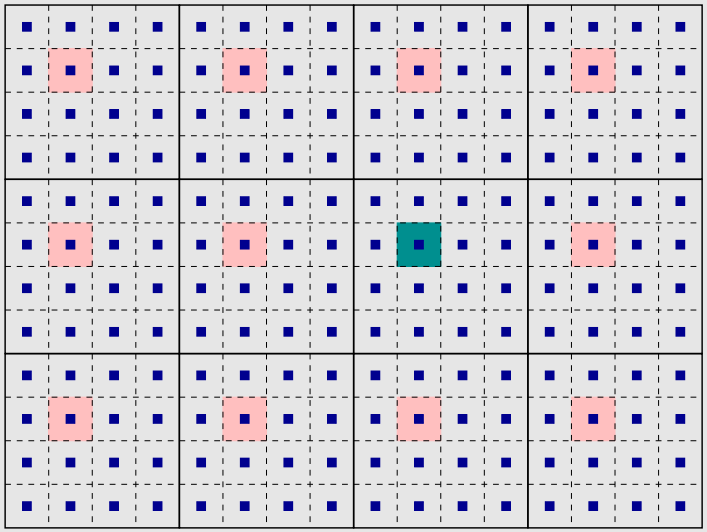

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$$P_{c,e} \leq \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \mathrm{Vol}(\Lambda_b) \sum_{\mathbf{r}\in\Lambda_e} e^{-\frac{\|\mathbf{r}\|^2}{2\sigma^2}}$$

$$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \mathrm{Vol}(\Lambda_b)\,\Theta_{\Lambda_e}\left(\frac{1}{2\pi\sigma^2}\right)$$

where

$$\Theta_\Lambda(y) = \sum_{\vec{x}\in\Lambda} q^{\|\vec{x}\|^2}, q = e^{-\pi y}, y > 0$$

is the **theta series** of Λ.

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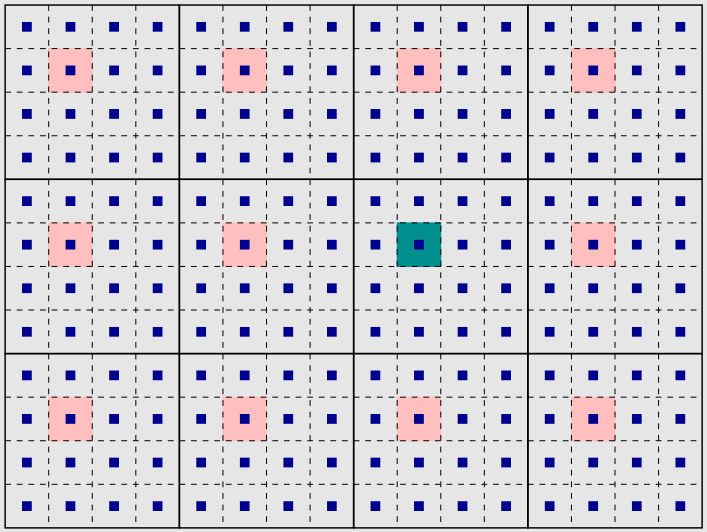

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Problem

Find Λ minimizing $\Theta_\Lambda(y)$.

Secrecy function

Definition

Let Λ be a $n-$dimensional lattice with fundamental volume λ^n. Its **secrecy function** is defined as,

$$\Xi_\Lambda(y) \triangleq \frac{\Theta_{\lambda\mathbb{Z}^n}(y)}{\Theta_\Lambda(y)} = \frac{\vartheta_3^n\left(e^{-\pi\sqrt{\lambda}y}\right)}{\Theta_\Lambda(y)}$$

where $\vartheta_3(q) = \sum_{n=-\infty}^{+\infty} q^{n^2}$ and $y > 0$.

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Examples

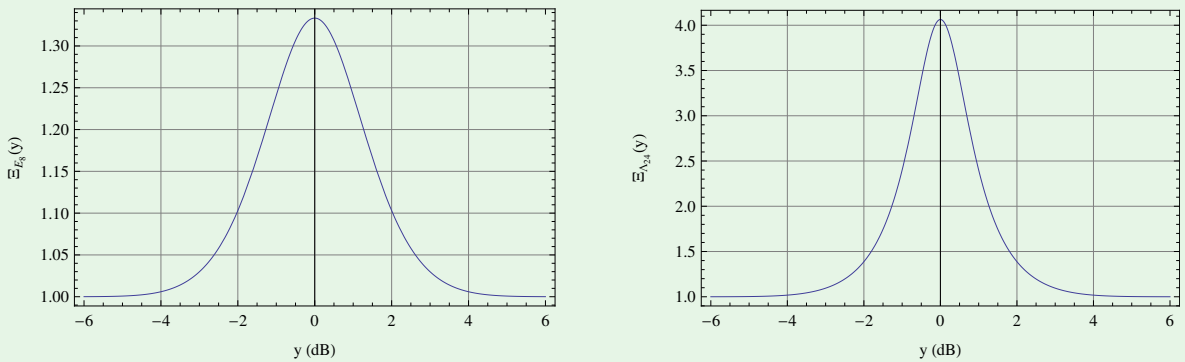

Figure : Secrecy functions of E_8 and Λ_{24}

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- A lattice equivalent to its dual has a theta series with a **multiplicative symmetry point** at $d(\Lambda)^{-\frac{1}{n}}$ (**Poisson-Jacobi's** formula),

$$\Xi_\Lambda\left(d(\Lambda)^{-\frac{1}{n}}\, y\right) = \Xi_\Lambda\left(\frac{d(\Lambda)^{-\frac{1}{n}}}{y}\right)$$

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Definition

For a lattice Λ equivalent to its dual and of determinant (volume) $d(\Lambda)$, we define the **weak secrecy gain**,

$$\chi_\Lambda \triangleq \Xi_\Lambda\left(d(\Lambda)^{-\frac{1}{n}}\right)$$

Conjecture

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If Λ is a unimodular lattice, then the strong and the weak secrecy gains coincide.

Corollary

The strong secrecy gain of a unimodular lattice Λ is $\chi^s_\Lambda \triangleq \Xi_\Lambda(1)$.

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The strong secrecy gain of a unimodular lattice Λ is $\chi_\Lambda^s \triangleq \Xi_\Lambda(1)$.

Calculation of E_8 secrecy gain

From E_8 theta series,

$$\frac{1}{\Xi_{E_8}(1)} = \frac{\frac{1}{2}\left(\vartheta_2(e^{-\pi})^8 + \vartheta_3(e^{-\pi})^8 + \vartheta_4(e^{-\pi})^8\right)}{\vartheta_3(e^{-\pi})^8}$$

$$= \frac{3}{4} \quad \left(\text{since } \frac{\vartheta_2\left(e^{-\pi}\right)}{\vartheta_3\left(e^{-\pi}\right)} = \frac{\vartheta_4\left(e^{-\pi}\right)}{\vartheta_3\left(e^{-\pi}\right)} = \frac{1}{\sqrt[4]{2}}\right)$$

so we get $\boxed{\chi_{E_8} = \Xi_{E_8}(1) = \frac{4}{3}}$.

Part III

Even Unimodular Lattices

Outline

Even Unimodular Lattices

Example

E_8 or the Leech lattice Λ_{24} are even unimodular.

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Properties

An even unimodular lattice Λ only exists when n is a multiple of 8. The minimum squared length of any non zero vector is upperbounded

$$\delta^2 \leq 2(m+1)$$

where $n = 24m + 8k$, $k = 0, 1, 2$. A lattice achieving this upperbound is called **extremal**. E_8 and Leech lattice are extremal.

Secrecy Gain of Extremal Lattices

Secrecy Functions in dimensions 72 and 80

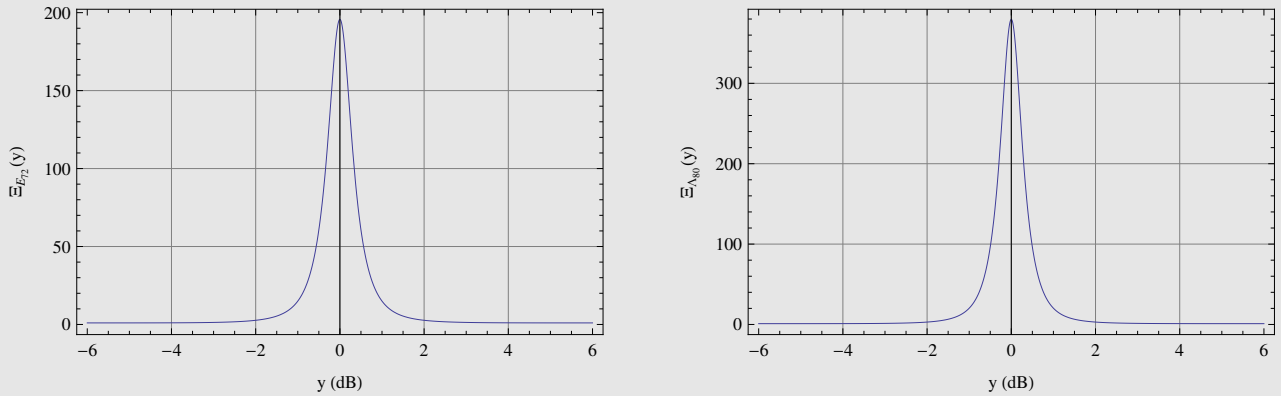

Figure : Secrecy functions of extremal lattices ($n = 72, 80$)

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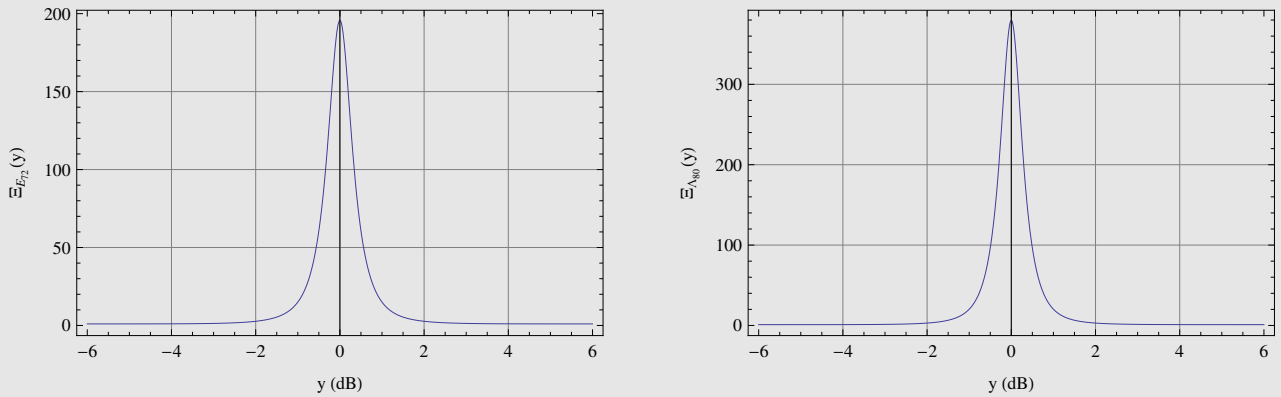

Figure : Secrecy functions of extremal lattices ($n = 72, 80$)

Secrecy gains of extremal lattices (all rational numbers !!!)

Dimension	8	24	32	48	72	80
Secrecy gain	$\frac{4}{3}$	$\frac{256}{63}$	$\frac{64}{9}$	$\frac{524288}{19467}$	$\frac{134217728}{685881} \simeq 195.7$	$\frac{536870912}{1414413} \simeq 380$

Secrecy Gain of extremal Even Unimodular Lattices

Theorem

The secrecy gain of an extremal even unimodular lattice is a rational number.

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Proof.

Theta series of an even unimodular lattice Λ ($n = 24m + 8k$),

$$\Theta_\Lambda = \sum_{j=0}^{m} b_j E_4^{3(m-j)+k} \Delta^j$$

with $E_4 = \frac{1}{2}\left(\vartheta_2^8 + \vartheta_3^8 + \vartheta_4^8\right)$, $\Delta = \frac{1}{256}\left(\vartheta_2\vartheta_3\vartheta_4\right)^8$ and $b_j \in \mathbb{C}$. As the lattice is extremal, b_j are computed by solving a linear system with integer coefficients, so $b_j \in \mathbb{Q}$. As

$$\begin{cases} \vartheta_2\left(e^{-\pi}\right) &= \vartheta_4\left(e^{-\pi}\right) \\ \vartheta_3\left(e^{-\pi}\right) &= \sqrt[4]{2}\vartheta_4\left(e^{-\pi}\right) \end{cases},$$

we obtain

$$E_4\left(e^{-\pi}\right) = \frac{3}{4}\vartheta_3^8\left(e^{-\pi}\right) \qquad \text{and} \qquad \Delta\left(e^{-\pi}\right) = \frac{1}{2^{12}}\vartheta_3^{24}\left(e^{-\pi}\right)$$

giving the rationality of $\Xi_\Lambda(1)$. □

Asymptotic behavior (I)

- Want to study the behavior of even unimodular lattices when $n \to \infty$.

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First answer

Apply the Siegel-Weil formula,

$$\sum_{\Lambda \in \Omega_n} \frac{\Theta_\Lambda(q)}{|\mathrm{Aut}(\Lambda)|} = M_n \cdot E_k\left(q^2\right)$$

where

$$M_n = \sum_{\Lambda \in \Omega_n} \frac{1}{|\mathrm{Aut}(\Lambda)|}$$

and E_k is the Eisenstein series with weight $k = \frac{n}{2}$. Ω_n is the set of all inequivalent $n-$dimensional, even unimodular lattices. We get

$$\Theta_{n,\mathrm{opt}}\left(e^{-\pi}\right) \leq E_k\left(e^{-2\pi}\right)$$

Asymptotic behavior (II)

Maximal Secrecy gain

For a given dimension n, multiple of 8, there **exists** an even unimodular lattice whose secrecy gain is

$$\chi_n \geq \frac{\vartheta_3^n\left(e^{-\pi}\right)}{E_k\left(e^{-2\pi}\right)} \simeq \frac{1}{2}\left(\frac{\pi^{\frac{1}{4}}}{\Gamma\left(\frac{3}{4}\right)}\right)^n \simeq \frac{1.086^n}{2}$$

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Behavior of Eisenstein Series

We have

$$E_k\left(e^{-2\pi}\right) = 1 + \frac{2k}{|B_k|}\sum_{m=1}^{+\infty}\frac{m^{k-1}}{e^{2\pi m}-1}$$

B_k being the Bernoulli numbers. For k a multiple of 4, then $E_k\left(e^{-2\pi}\right)$ fastly converges to 2 ($k \to \infty$).

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Bound from Siegel-Weil Formula vs. Extremal lattices

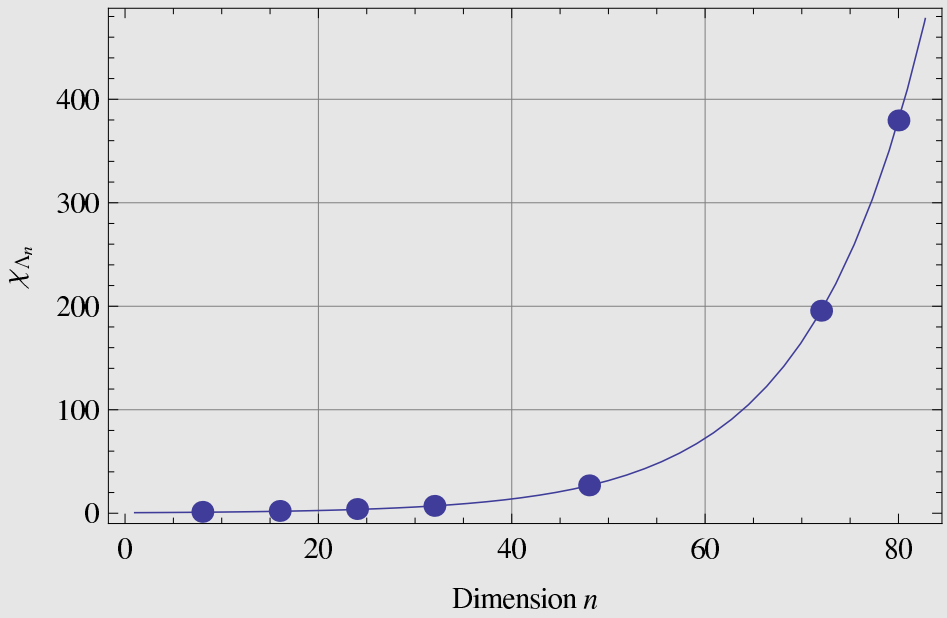

Figure : Lower bound of the minimal secrecy gain as a function of n from Siegel-Weil formula. **Points** correspond to **extremal lattices**.

Another way of analyzing the asymptotic behavior

Expression of the theta series

For a $2k-$dimensional even unimodular lattice, the Fourier decomposition gives

$$\Theta_\Lambda(z) = E_k(z) + S_k(z,\Lambda) = \sum_{m=0}^{\infty} r(m,\Lambda)\, e^{2i\pi mz}$$

where $S_k(z,\Lambda)$ is a cusp form.

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Fourier coefficients

If $S_k(z,\Lambda) = \sum_{m=0}^{\infty} a(m,\Lambda)\, e^{2i\pi mz}$, then,

$$r(m,\Lambda) = \underbrace{\frac{(2\pi)^k}{\zeta(k)\Gamma(k)}\sigma_{k-1}(m)}_{E_k} + \underbrace{a(m,\Lambda)}_{S_k}$$

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Asymptotics

Asymptotic analysis gives

$$\begin{cases} \sigma_{k-1}(m) & = O\left(m^{k-1}\right) \\ a(m,\Lambda) & = O\left(m^{\frac{k}{2}}\right) \end{cases}$$

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Conclusion

Coefficients of E_k are asymptotic estimates of the coefficients of Θ_Λ. The secrecy gain of any even unimodular lattice behaves like

$$\boxed{\frac{\vartheta_3^{2k}\left(e^{-\pi}\right)}{E_k\left(e^{-2\pi}\right)}}$$

when $k \to \infty$.

Outline

Flatness Factor

Information Leakage

Let M be the transmitted secret message and Z^n be the vector received by Eve. Then,

$$I\left(\mathsf{M};\mathsf{Z}^n\right) \leq 2\varepsilon_{\Lambda_n}(\sigma)\left(nR - \log\varepsilon_{\Lambda_n}(\sigma)\right)$$

where

$$\varepsilon_{\Lambda_n}(\sigma) = \left(\frac{\mathrm{Vol}\left(\Lambda_n\right)^{\frac{2}{n}}}{2\pi\sigma^2}\right)^{\frac{n}{2}} \Theta_{\Lambda_n}\left(\frac{1}{2\pi\sigma^2}\right) - 1$$

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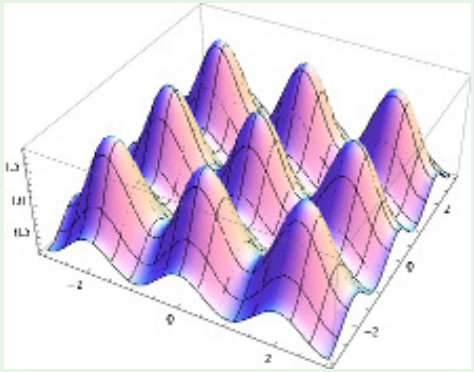
Remark

True definition of the flatness factor is

$$\varepsilon_{\Lambda_n}(\sigma) = \max_{\boldsymbol{x}\in\mathcal{V}(\Lambda_n)} \left| \frac{\sum_{\boldsymbol{t}\in\Lambda_n}\left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} e^{-\frac{\|\boldsymbol{x}-\boldsymbol{t}\|^2}{2\sigma^2}}}{1/\mathrm{vol}\left(\Lambda_n\right)} - 1 \right|.$$

Intuition



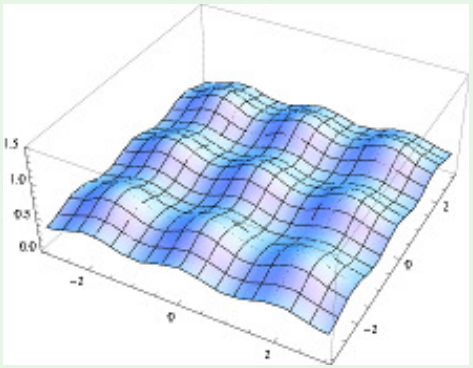


Figure : Sum of Gaussian Measures on the $2\mathbb{Z}^2$ lattice with $\sigma^2 = 0.3$ and $\sigma^2 = 0.6$

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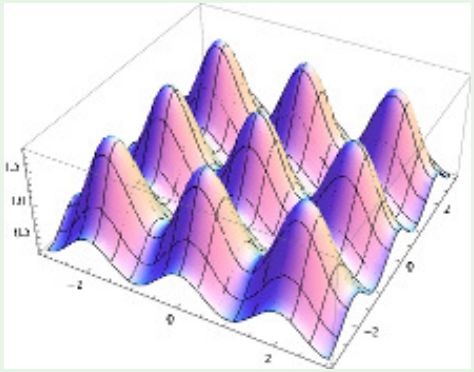

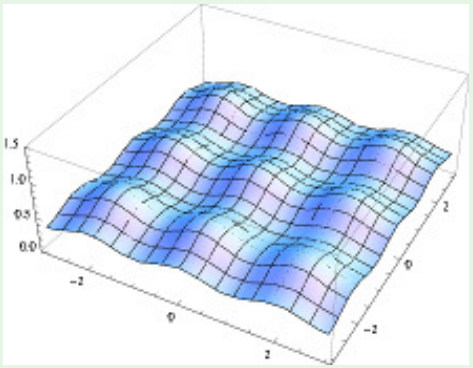

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Goal

Is it possible to obtain a vanishing flatness factor? For which values of σ?

Asymptotics of the flatness factor

Flatness factor of an even unimodular lattice

For n large enough, randomly choose an even unimodular lattice Λ_n. Then, set $y = \frac{1}{2\pi\sigma^2}$ (and $k = \frac{n}{2}$),

$$\begin{aligned} \varepsilon_{\Lambda_n}(\sigma) &= y^{\frac{n}{2}} \Theta_{\Lambda_n}(iy) - 1 \\ &\simeq y^k E_k(iy) - 1 \\ &\simeq y^k \end{aligned}$$

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Strong secrecy for even unimodular lattices

We thus get

$$\varepsilon_{\Lambda_n}(\sigma) \underset{n\to\infty}{\to} \begin{cases} 0 & \sigma^2 > \frac{1}{2\pi} \to \text{strong secrecy} \\ 1 & \sigma^2 = \frac{1}{2\pi} \\ \infty & \sigma^2 < \frac{1}{2\pi} \end{cases}$$

Outline

Construction A

Construction A using $\mathbb{Z}$

Let q be an integer. Then, $\mathbb{Z}/q\mathbb{Z}$ is a finite field if q is a prime and a finite ring otherwise. For a linear code $\mathscr{C}$ of length n defined on $\mathbb{Z}/q\mathbb{Z}$, lattice Λ is given by

$$\Lambda = q\mathbb{Z}^n + \mathscr{C} \triangleq \bigcup_{\boldsymbol{x}\in\mathscr{C}} \left(q\mathbb{Z}^n + \boldsymbol{x}\right).$$

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Construction of D_4

D_4 is obtained as

$$D_4 = 2\mathbb{Z}^4 + (4,3,2)_{\mathbb{F}_2} = (1+i)\mathbb{Z}[i]^2 + (2,1,2)_{\mathbb{F}_2}$$

where $(4,3,2)_{\mathbb{F}_2}$ is the binary parity-check code.

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Construction of E_8

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$$E_8 = 2\mathbb{Z}^8 + (8,4,4)_{\mathbb{F}_2} = \bigcup_{\boldsymbol{x}\in(8,4)_{\mathbb{F}_2}} \left(2\mathbb{Z}^8 + \boldsymbol{x}\right)$$

where $(8,4,4)_{\mathbb{F}_2}$ is the extended binary Hamming code $(7,4,3)_{\mathbb{F}_2}$.

Construction A (quaternary)

Construction A of the Leech lattice

The **Leech lattice** can be obtained as

$$\Lambda_{24} = 4\mathbb{Z}^{24} + (24, 12)_{\mathbb{Z}_4}$$

where $(24, 12)_{\mathbb{Z}_4}$ is the quaternary self-dual code obtained by extending the quaternary cyclic Golay code over $\mathbb{Z}_4$.

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Other constructions

Construction A can be generalized. Constructions B, C, D or E for instance. But one can show that all these constructions are equivalent to construction A with a suitable alphabet.

Constructions with codes

Binary construction A

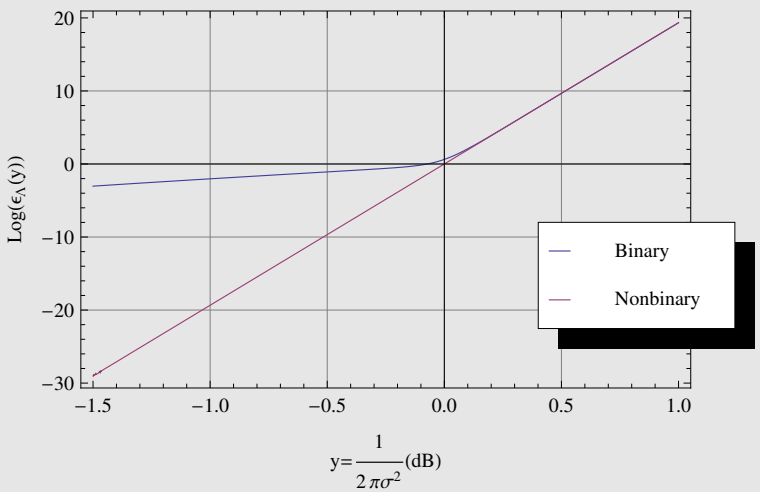

Figure : Even Unimodular Lattice in dimension 168: binary **vs** general case

Constructions with codes

Binary construction A

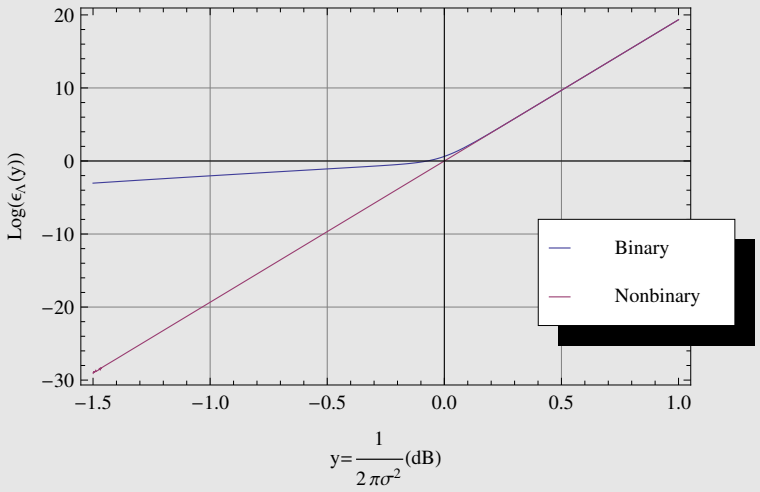

Figure : Even Unimodular Lattice in dimension 168: binary **vs** general case

Binary lattices: a set of negligible measure

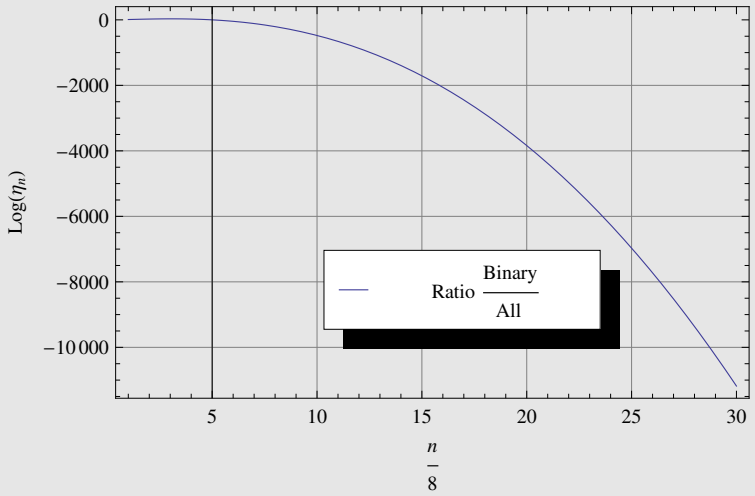

Figure : Binary Even Unimodular Lattice : How many they are?